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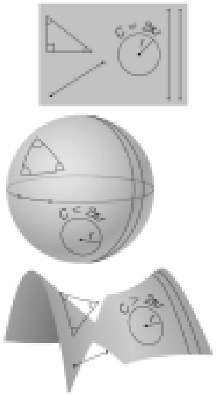
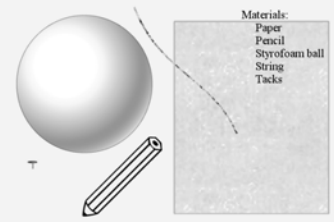


Envisioning the Curvature of Space

Learning Objective

Upon completion of this activity, the student will be able to summarize what is meant by the curvature of space; the difference between a flat, curved, and an open universe; and attempt to describe the curvature of spacetime[i].



Materials

- Paper
- Pencil
- Styrofoam ball
- String
- Tacks

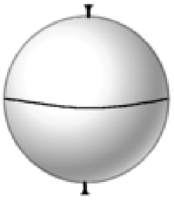
Introduction

> *Like a great new sculpture, so curvature preserves its unity, strength, and beauty as we detect and measure it from three quite different standpoints, called* ***bending****,* ***transport****, and* ***distance****. This unity of different views of curvature we see most vividly in everyday examples of curved surfaces. Here the geometry has only two dimensions, not four. And here the medium is not spacetime, but space. It matters little whether the background for our thinking is a saddle or a boulder, a planet or a potato, a Zeppelin or a ball. So, for definiteness—and simplicity—let's consider the two-dimensional surface of a ball*[ii].

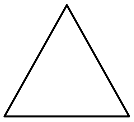
Procedure

Bending

Take two tacks and place them on precisely opposite ends of the ball so that you have defined north and south. Using a piece of string or pen, mark a geodesic (a great circle) halfway in between the N and S poles to represent the "equator."

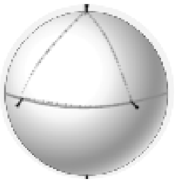


A. What does the sum of the interior angles of a triangle always equal? On the flat sheet of paper, draw an equilateral triangle. How large is each interior angle in degrees? How large in radians? [Note: 1 degree = 0.0174532925 radians; there are 2 π radians in 360°]



Sum (degrees)__________ Angle size (degrees)__________ Angle size (radians)__________

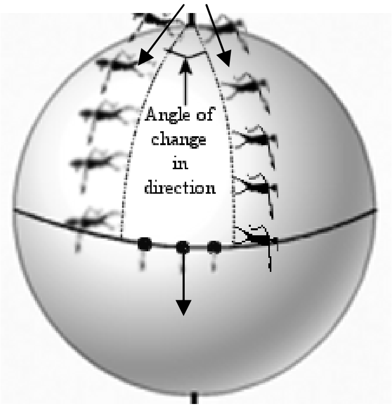
B. Using two more tacks and the string, make an equilateral triangle on the ball (see figure at the right). How many of these triangles could you make around the north pole of the sphere if none of them overlapped? How large is the angle at the N pole for one of the triangles in degrees? In radians? Since this is an equilateral triangle by design in curved space, what is the sum of the angles?



How many?__________ Angle size at N pole (degrees)__________ Angle at NP (radians)__________

Sum of the angles?__________

Transport



In this picture we see a person who always points due south. Walking from the N pole, he reaches the equator and immediately turns 90 degrees and walks some random distance due east, still pointing due south. After walking for a long while, he turns due north, but still points due south. The pointing direction is always directly south, and thus the person's arm has not changed its direction angle at all, not one iota. This is called *parallel transport*.

A. However, when the person reaches the N pole after his trip, is the arm turned relative to its start-of direction? What is that angle of change in direction equal to? That is, is it a) greater than, b) equal to, or c) less than the angle "swept out" due to the distance the person traveled while at the equator?

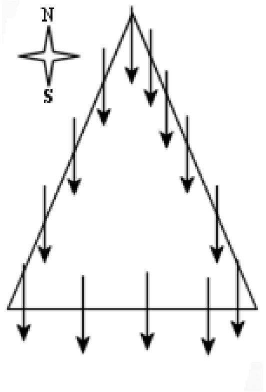
Answer: ____ a) ____b) ____c)

B. Can you calculate the space curvature for this example? To find the change in direction and the space area circumnavigated, you need to figure out how much of the surface area of the sphere is enclosed by the person starting at the N pole, walking to the equator, turning 90 degrees east, walking a distance, turning 90 degrees north, and walking back to the N pole. Let's say that this person walks 1/6 of the way around the sphere at the equator. What is the angle change in direction that corresponds to that distance? Take 360 degrees and multiply it by 1/6 and you get 60 degrees. What fraction is this area of the total area of the sphere? Answer: 1/12. How was that determined?

$$\text{space curvature} = \frac{\text{change in direction during parallel transport}}{\text{space area circumnavigated}}$$

Curvature = _________________ radians per square _____ (meter or centimeter)

C. Consider this person on a flat plane, always pointing same direction, represented in this figure by all of the arrows pointing south, and then coming back to the starting point. How is this final angle compared to the initial angle?



________Final angle is greater than the initial angle.

________Final angle is equal to the initial angle

________Final angle is less than the initial angle.

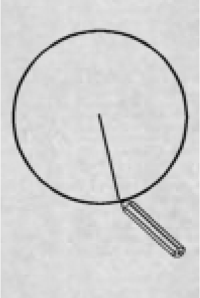
Distance

"Discrepancy in circumference reveals curvature!"[iii]

A. Run string around the equator of the ball, and measure the circumference as accurately as you can. What is the circumference? _________ (This would be the same number no matter which great circle—geodesic—you used.)

B. Divide the string in half twice so that it is one-quarter of the length it was. What is this shorter length? _____

Fasten one end of this shortened string at the N pole, and loop a pencil through the other end. Draw a circle around the ball. It should match up exactly with the equator, right? You could use this shortened length of string to draw great circles all over the ball.



C. Now, take that shortened length of string, and your pencil, and draw a circle on a flat piece of paper. You will get a circumference given by the equation: $circum = 2\pi \times radius$.

What is the circumference of this circle? ________

D. What is the difference in the circumferences between this circle and the great circle on the ball? _________

What is the ratio of the circumferences? (This is the fractional circumference discrepancy, divide the sphere circumference by the circle circumference.)?________

E. Cut out the circle from your flat piece of paper. Mark the very center and pin it to the ball at the N pole. What would you have to do to get the circle to fit tightly over the ball? You can draw a picture if it helps to explain this.

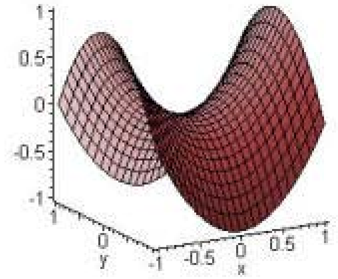
F. We can estimate the space curvature from this "experiment," however, by working with just the sphere and the flat paper (the "space area responsible for discrepancy" is the area of the circle):

$$\text{space curvature} = \frac{\text{fractional circumference discrepancy}}{\text{space area responsible for discrepancy}} \times 4\pi$$

Space curvature = _________________ radians per square _____ (meter or centimeter)

How does your value for the space curvature calculated here compare to your previous calculation under *parallel transport*?

It is hard to create a negatively (hyperbolic) curved space (saddle), and thus we will just state what the comparison of the circumference of a circle on the flat piece of paper and on a negatively curved space would give us: The negative curvature of this saddle would give us a longer circumference of a circle (given the same radius) than that of the flat space. While we keep that string taut when drawing our circle, we have to move up and down the saddle, and this lengthens the circumference. [This seems to be a concept used often in the design of amusement park rides.]



Questions

1. What is meant by “the curvature of space”?

2. How do these three curvature-of-space examples relate to the curvature of the Universe?

3. Imagine a way to include the curvature of spacetime in this activity, or even just the curvature of time. What do you think? If you had all of the resources on Earth available to you, what would you do?

4. What was the sum of the interior angles of the triangle on the positively curved sphere when you considered a 60-degree angle of change in direction? Considering your answer for “Bending, Part B,” what can be accurately stated for the sums of the interior angles of triangles in positively curved space?
    a. The sum of the angles equals 270 degrees.
    b. The sum of the angles is greater than 180 degrees.
    c. The sum of the angles is equal to 180 degrees
    d. The sum of the angles is less than 180 degrees.

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[i] Information and ideas taken from Wheeler, J.A., A Journey into Gravity and Spacetime (Scientific American Library, 1999), pp. 78-81

[ii] *Ibid.* p. 76

[iii] *Ibid.* p. 81