

Carroll & Ostlie “An Introduction to Modern Astrophysics” 2nd Ed.

Problem 14.13, p. 516

14.13 In this problem you will carry out a nonlinear calculation of the radial pulsation of the one-zone model described in Example 14.3.1 (see handout). The equations that describe the oscillation of this model star are Newton’s second law for the forces on the shell,

$$m \frac{dv}{dt} = -G \frac{Mm}{R^2} + 4\pi R^2 P, \quad (14.20)$$

and the definition of the velocity, v , of the mass shell,

$$v = \frac{dR}{dt}. \quad (14.21)$$

As in Example 14.3.1, we assume that the expansion and contraction of the gas are adiabatic:

$$P_i V_i^\gamma = P_f V_f^\gamma, \quad (14.22)$$

where the “initial” and “final” subscripts refer to any two instants during the pulsation cycle.

(a) Explain (in words) the meaning of each term in Eq. (14.20).

(b) Use Eq. (14.22) to show that

$$P_i R_i^{3\gamma} = P_f R_f^{3\gamma}. \quad (14.23)$$

[Hint: Use definition of volume and solve for P_f in terms of P_i , R_i , R_f .]

(c) You will not be taking derivatives. Instead, you will take the difference between the initial and final value of the radius R and radial velocity v of the shell divided by the time interval Δt separating the initial and final values. That is, you will use $(v_f - v_i)/\Delta t$ instead of dv/dt , and $(R_f - R_i)/\Delta t$ instead of dR/dt in Eqs. (14.20) and 14.21. A careful analysis shows that you should use $R = R_i$ and $P = P_i$ on the right-hand side of Eq. (14.20), and use $v = v_f$ on the left-hand side of Eq. (14.21). Make these substitutions in Eqs. (14.20) and (14.21) and show that you can write

$$v_f = v_i + \left(\frac{4\pi R_i^2 P_i}{m} - \frac{GM}{R_i^2} \right) \Delta t \quad (14.24)$$

and

$$R_f = R_i + v_f \Delta t. \quad (14.25)$$

(d) Now you are ready to calculate the oscillation of the model star. The mass of a typical classical Cepheid is $M = 1 \times 10^{31}$ kg ($5 M_{\odot}$), and the mass of the surface layers may be arbitrarily assigned $m = 1 \times 10^{26}$ kg. For starting values at time $t = 0$, take

$$\begin{aligned} R_i &= 1.7 \times 10^{10} \text{ m} \\ v_i &= 0 \text{ m s}^{-1} \\ P_i &= 5.6 \times 10^4 \text{ N m}^{-2} \end{aligned}$$

and use a time interval of $\Delta t = 10^4$ s. Take the ratio of specific heats to be $\gamma = \frac{5}{3}$ for an ideal monatomic gas. Use Eq. (14.24) to calculate the final velocity v_f at the end of one time interval (at time $t = 1 \times 10^4$ s); then use Eq. (14.25) to calculate the final radius R_f and Eq. (14.23) to calculate the final pressure P_f . Now take these final values to be your new initial values, and find new values for R , v , and P after two time intervals (at a time $t = 2 \times 10^4$ s). Continue to find R , v , and P for 150 time intervals, until $t = 1.5 \times 10^6$ s. Make three graphs of your results: R vs t , v vs t , and P vs t . Plot the time on the horizontal axis.

(e) From your graphs, measure the period Π of the oscillation (both in seconds and in days) and the equilibrium radius, R_0 , of the model star. Compare this value of the period with that obtained from Eq. (14.14):

$$\Pi = \frac{2\pi}{\sqrt{\frac{4}{3}\pi G \rho_0 (3\gamma - 4)}} \quad (14.14)$$

where $\rho_0 = \frac{M}{\frac{4}{3}\pi R_0^3}$ is the average density of the equilibrium model (assume an ideal monatomic

gas). Also compare your results with the period and radial velocity observed for δ Cephei.

(f) [5-pt bonus] Fold the velocity versus time data so that you have a graph that shows the velocity phased over 1.5 or even 2 cycles of the period you find (see example at the right for an orbital phase). Does your graph look right? What would this graph look like if you didn't have the correct period?

