# ASTR 323 Spring 2009

## Final exam Solution

1. (25 pt) The Big Rip

a) (4 pt) **Explain** what is meant by the scale factor, $R(t)$.

Solution: The scale factor is a dimensionless parameter that measures the expansion rate of the metric of the Universe. The instantaneous distance between any two points in the Universe, $r(t)$, scales as $r(t) = R(t)\varpi$ where $\varpi$ is the separation of those two points at the present time, $t_0$. Thus, $R(t_0) = 1$. Since we know the Universe is expanding, $R(t) < 1$ for $t < t_0$ and $R(t) > 1$ for $t > t_0$. The Hubble parameter is defined as $H(t) = \dot{R}/R$ and the Hubble constant as $H_0 = H(t_0) = \dot{R}(t_0)$.

b) (4 pt) Suppose you live in a flat Universe that is dominated by dark energy which has a density which is not constant with time, but has an equation of state such that $\rho_{DE} = \rho_{DE,0}R^3$. Replacing $\rho_\Lambda$ by $\rho_{DE}$ in the Friedmann equation, and setting $\rho_r = \rho_m = 0$ (dark-energy dominates), **write down the differential equation which governs** $R(t)$**.**

Solution: This model is sometimes referred to as the "phantom energy" model. Replacing $\rho_\Lambda$ by $\rho_{DE,0}R^3$ in the Friedmann equation, it becomes $\dot{R}^2 = (8\pi G)\rho_{DE,0}R^5/3$. Noting that $H_0 = \dot{R}(t_0)/R(t_0) = \sqrt{8\pi G/3\rho_{DE,0}}$, and taking square root of the Friedmann equation, I find: $\boxed{dR/dt = H_0 R^{5/2}}$. Note that I have chosen the positive sign in taking the square root since we know that the Universe is expanding; were it contracting I would have chosen the negative sign.

c) (7 pt) Put all factors of $dR$ and $R$ on one side of this equation and all factors of $dt$ and $t$ on the other side, and using the fact that $\int dRR^{-5/2} = -\frac{2}{3}R^{-3/2}$, **integrate** the equation from part (b) from $R = 1$ to $R_f$ and from $t_0$ to $t_f$. Then **express** $t_f$ in terms of $H_0$, $t_0$ and $R_f$.

Solution: Okay, so $dRR^{-5/2} = H_0 dt$. Integrating this, $\frac{2}{3}\left(1 - R_f^{-3/2}\right) = H_0(t_f - t_0)$. Solving for $t_f$: $\boxed{t_f = t_0 + \frac{2}{3H_0}\left(1 - R_f^{-3/2}\right)}$.

d) (6 pt) Setting $R_f = \infty$ in this equation, **show that** $t_f$ is a finite time; in other words the scale factor of the Universe becomes infinite in a finite amount of time. Taking $H_0 = (14\text{Gyr})^{-1}$, **calculate** how much time the Universe has until it is infinitely large. **What does this imply** about the fate of the Universe (hint: see the title of this problem)?

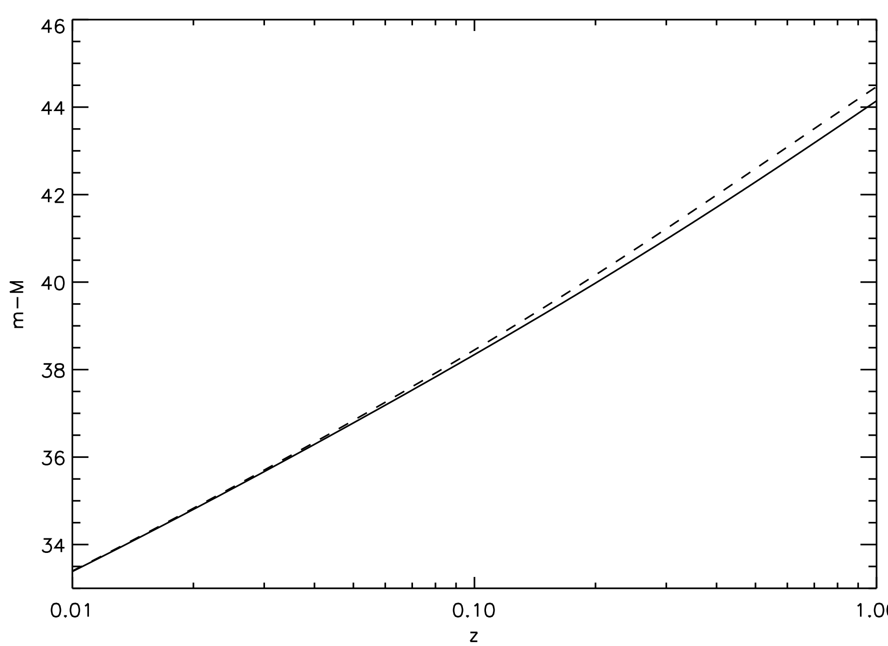
Solution: Setting $R_f = \infty$, this equation becomes $t_f = t_0 + \frac{2}{3H_0}$. Since $H_0$ is non-zero, this equation is finite. Plugging in $H_0 = (14\ \text{Gyr})^{-1}$ (note that there was a typo on the exam as I forgot to put the parenthesis around 14 Gyr - I have taken this into account in grading the exam), I find $\boxed{t_f = 9.3\ \text{Gyr} + t_0}$. What happens at that point in time is described in more detail here: http://arxiv.org/abs/astro-ph/0302506

Basically, phantom energy causes the Universe to expand to infinite dimensions in a finite amount of time, at which point galaxies, stars, and even atoms and nuclei are ripped apart.

e) (4 pt) **Explain** how one might distinguish this model from dark energy, $\rho_{DE} = \rho_{DE,0} R^3$ from the model we discussed in class with $\rho_\Lambda$ = constant, using observations of the fluxes of Type Ia supernovae as a function of redshift (as we know these are standard candles).

Solution: The luminosity distance (or distance modulus) versus redshift (i.e. the Hubble diagram) depends on the expansion rate of the Universe, so a different behavior for dark energy will change the expansion rate. Since $R = 1/(1+z)$, as we go to higher redshift, this model for dark energy becomes smaller in density than the matter energy density much more quickly than the cosmological constant ($\Lambda$) model. Thus, a different model for dark energy will cause a different dependence of luminosity distance on redshift which may be detected by measuring the luminosity distances and redshifts of Type Ia supernovae.

Extended answer: The Hubble parameter scales as $H = H_0(\Omega_{m,0} R^3 + \Omega_{DE,0} R^{-3})$, so in this model the Hubble parameter becomes smaller more quickly in the past. Now, looking backwards in time, $d\varpi = -c dt/R$, but $dt/R = -dz/H$, so $d\varpi = c dz/H$. Since $H$ gets smaller faster with redshift in the phantom model, then $\varpi$ grows more quickly with redshift, causing galaxies to appear fainter than in the cosmological constant model. See the following plot for a comparison of the phantom model (dashed line) to the cosmological constant model (solid line) for $H_0 = 70$ km/s/Mpc, $\Omega_{m,0} = 0.27$, and $\Omega_\Lambda = \Omega_{DE,0} = 0.73$. The Type Ia supernovae in the phantom model are about 0.3 magnitudes fainter above a redshift of 0.4 compared to the cosmological constant model.



2. (19 pt) Quasar mass and Eddington limit

a) (6 pt) As discussed in class, virial mass estimates use the balance between orbital motion and gravitational pull to derive the enclosed mass of a system. Using either dimensional analysis or physical arguments, **derive the enclosed mass,** $M$ in terms of the velocity $v$ of a small object on a circular orbit at radius $r$, assuming spherical symmetry.

Solution: Balancing centripetal acceleration and gravitational attraction, $v^2/r - GM/r^2 = 0$, so $\boxed{M = v^2 r/G}$.

b) (4 pt) Broad emission lines from a quasar have a velocity dispersion of 3000 km/s and a scale of $10^{15}$ m as determined by reverberation mapping. **Estimate the mass** of the black hole in solar mass units using your result from part (a), and assume the velocity dispersion corresponds to clouds on circular orbits.

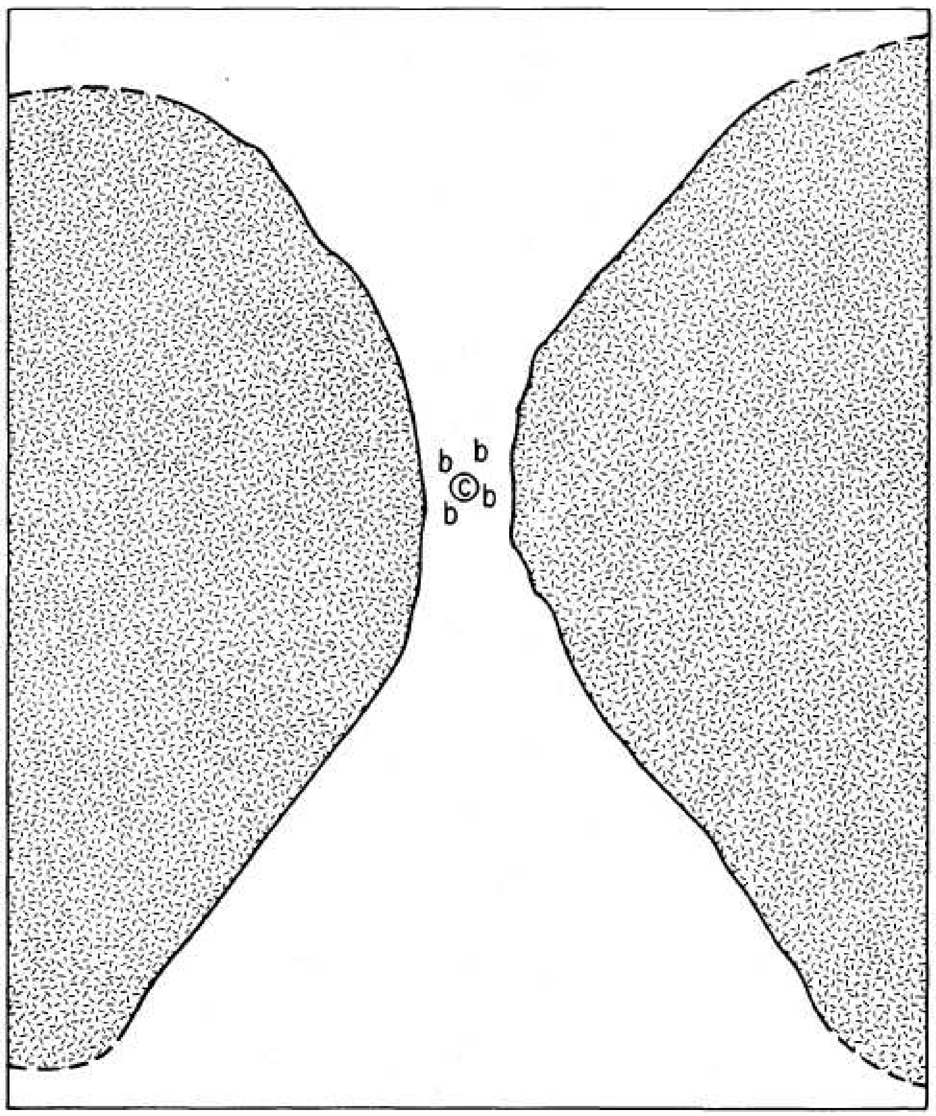
Solution: Using this mass estimator, $M = (3000\text{km/s})^2 10^{15} m / 6.67\times10^{-11}\text{m}^3\text{s}^{-2}\text{kg}^{-1} = 1.3 \times 10^{38}\text{kg}$. Converting this to solar mass units, $\boxed{M = 6.7 \times 10^7 M_\odot}$. For those of you that used $10^{15}$ km you should have obtained an answer $10^3$ times larger.

c) (4 pt) The quasar is seen to have a luminosity of $10^{38}$ Watts. **What is the ratio** of this luminosity to the Eddington limit? (If you were unable to complete part (b), use a black hole mass of $10^8 M_\odot$.)

Solution: The Eddington luminosity for a black hole of this mass is $L_{Edd} = 8.8 \times 110^{38}$ W. So, the ratio of the observed luminosity to the Eddington luminosity is $\boxed{L/L_{Edd} = 11.4\%}$. Those of you that found a black hole mass using km rather than m should find an Eddington ratio 1000 times smaller.

d) (5 pt) As this quasar shows broad emission lines, **are these lines** permitted, forbidden, or both? **Can we see** the accretion disk, or is it obscured by the dust torus? **Explain** your reasoning using a labeled diagram.

Solution: Broad emission lines are emitted from dense clouds that are close to the accretion disk. At high densities atoms that are excited by photoionization by the accretion disk have a higher probability of de-exciting via collisions than via emission of a photon. Consequently, only permitted lines are seen from the broad-line region. Both the accretion disk and broad line region are obscured by the dusty torus, so if we can see the broad lines, we can see the disk, so the disk is not obscured by the torus. In the following diagram from the Antonucci & Miller (1985, ApJ, 297, 621), the disk continuum ('c') and broad-line region ('b') are both obscured by the dust torus (dotted region). So, this quasar is a Type I AGN which is viewed along the dust-free lines of sight, from above or below in this diagram.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3. (28 pt) Galaxy cluster

a) (4 pt) A galaxy cluster has a temperature of $10^8$ K. **Estimate the energy and velocity** of each proton in the cluster using the Boltzmann constant $k_B$ and temperature.

Solution: Using dimensional analysis, the kinetic energy of the protons is $\frac{1}{2}m_p v^2 \sim k_B T$ since $k_B T$ has units of energy (Joules). Plugging in the constants from the first page, this gives: $\boxed{v = 1283 \text{ km/s}}$. Note that if you used the more precise expression $\frac{1}{2}m_p v^2 = \frac{3}{2}k_B T$ you would have found $v = 1572$ km/s. Note that the electrons can have a much higher velocity - indeed higher than the escape velocity of the cluster - but the charge attraction to the protons means that they will not escape.

b) (2 pt) Given this velocity of hydrogen atoms in the cluster, **what do you expect** the typical velocity of galaxies in the cluster to be?

Solution: I would expect the typical velocity of galaxies to be similar to that of the protons. In either case the velocities are supporting against gravity, so they should be comparable.

c) (4 pt) Hubble's constant is $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$. **At what distance** do you expect the velocity dispersion of the cluster to be 10% of the apparent recession velocity, $cz$, due to cosmological redshift. If you did not solve part a, use a velocity of 1000 km s$^{-1}$.

Solution: By dimensional analysis, Hubble's law is $cz = H_0 d$ (both sides have units of km/s). We wants $v = 0.1cz = 0.1H_0 d$. Solving for $\boxed{d = 10v/H_0 = 183 \text{ Mpc}}$ using $v = 1283$ km/s. This means that at this distance the finger-of-God effect becomes less prominent (it is only 10% of the spread in distance estimated by Hubble's law), thus using Hubble's law to estimate distances becomes more accurate (to 10%).

d) (5 pt) If the cluster has a size of $r = 1.5$ Mpc, **estimate the mass** of the cluster in solar masses, using your result from problem 2a.

Solution: As before, $M = (1283\text{km/s})^2 1.5\text{Mpc} 3\times10^{22}\text{m/Mpc}/6.67\times10^{-11}\text{m}^3\text{s}^{-2}\text{kg}^{-1} = 1.1\times10^{45}\text{kg}$. Converting this to solar mass units, $\boxed{M = 5.6\times10^{14}M_\odot}$. This is right around the typical mass of rich galaxy clusters.

e) (5 pt) If all of this mass were in hot gas, **compute the average density** (kg m$^{-3}$) of the cluster within 1.5 Mpc. Assuming that all of the gas is ionized hydrogen, **compute the electron number density**, $n_e$ (m$^{-3}$).

Solution: The average density is the mass divided by the volume, so $\rho = \frac{3\times1.1\times10^{45}kg}{4\pi(4.5\times10^{22})^3\text{m}^3}$ or $\boxed{\rho = 2.9\times10^{-24} \text{ kg m}^{-3}}$. Now, for hydrogen gas, most of the mass is in protons, while there is one electron per proton, so the electron number density is $\boxed{n_e = \rho/m_p = 1711 \text{ m}^{-3}}$. This, of course, is the average density, while actually the density is declining with radius.

f) (8 pt) The X-ray luminosity of clusters scales as

$$L_X = 1.5 \times 10^{10} L_\odot \left(\frac{n_e}{100\text{m}^{-3}}\right)^2 \left(\frac{T}{10^8\text{K}}\right)^{1/2} \left(\frac{R}{1\text{Mpc}}\right)^3 . \quad (1)$$

where $n_e$ is the average electron density and $R$ is the size. The observed luminosity of this cluster is $L_X = 10^{11} L_\odot$. **Is this luminosity consistent** with the average electron number density, $n_e$, you computed in part e? If not, **what explains** the discrepancy? (If you did not complete part e, use a value of $n_e = 1000$ m$^{-3}$.)

Solution: Plugging in the $n_e$ from part (e), $L_X = 1.5 \times 10^{10} L_\odot (1711/100)^2 (10^8/10^8)^{1/2} 1.5^3$ or $\boxed{L_X = 1.5 \times 10^{13} L_\odot}$. This is *much* higher than the observed luminosity of $10^{11} L_\odot$. The reason for the discrepancy is that most of the mass in the cluster is *not* in ionized gas - it is in dark matter. So, this is evidence for a dark matter component to the cluster. Of course to tighten up the estimate of dark matter one needs to be more careful with using a model for the density of gas as a function of radius, etc.

4. (28 pt) Milky Way

a) (8 pt) Consider a star precisely on the opposite side of the Milky Way from the Sun whose light is lensed by the Galactic Center black hole of mass $M$ (see diagram - note that the angle $\alpha$ is greatly exaggerated, by about $10^5$). Use the impulse approximation to calculate how much a photon is deflected by the Galactic Center black hole as follows: give the photon a mass $m$ to **compute the force**, $F_g$, on the photon at the distance of closest approach, $R_E$, using Newton's law of gravity. Assume a duration of this force of $\Delta t = 4R_E/c$, **calculate the change of momentum** in a photon as it passes the black hole, $\Delta p \sim F_g \Delta t$. Now assign the photon an initial momentum $p = mc$, and **calculate the deflection angle** as $\alpha = \Delta p/p$, in radians, in terms of the Schwarzschild radius, $R_s$, and $R_E$.

Solution: This problem is a simplified version of the lensing calculation you did on the homework. The magnitude of the gravitational force (which I told you to memorize for the exam) at closest approach is $\boxed{F_g = GMm/R_E^2}$. So, $\Delta p = \frac{GMm}{R_E^2}\frac{4R_E}{c}$, or $\boxed{\Delta p = \frac{4GMm}{R_E c}}$. Dividing this by $p$ gives $\alpha = \frac{4GM}{R_E c^2}$, or $\boxed{\alpha = \frac{2R_S}{R_E}}$.

b) (6 pt) Using the diagram, **express the Einstein radius,** $R_E$, for the Galactic center black hole in terms of $\alpha$ and $d_L$ using the small angle approximation. Then, eliminate $\alpha$ using part (b) to **solve for** $R_E$ in terms of $d_L$ and the Schwarzschild radius, $R_s$, of the black hole.

Solution: Examining the diagram, $d_L\alpha = 2R_E$ using the small-angle approximation. Plugging in the result for $\alpha$ from part (a), $2d_L R_S/R_E = 2R_E$. Solving for $R_E$, $\boxed{R_E = \sqrt{R_S d_L}}$, which agrees with the lensing equation when $d_S = 2d_L$. So the Einstein radius is just the geometric mean of the Schwarzschild radius and the lens distance.

c) (6 pt) **Argue** that this lensing of a star on the other side of the Galaxy is yet another means of measuring the mass of the Galactic center black hole. For an angular size of the Einstein radius, $R_E/d_L = \alpha/2$, of 1.4 arcseconds, **what value** of $R_E$ in meters does this correspond to (assume 200,000 arcseconds per radian and $d_L = 8$ kpc)? Using $R_S = 3000(M/M_\odot)$ m, **compute the mass** of the Galactic Center black hole.

Solution: If we know $d_L$, then we can determine $R_S$, and thus $M$ from the relation derived in section (b). Using $R_E/d_L = \alpha/2 = \frac{1.4''}{200{,}000''rad^{-1}} = \sqrt{R_S/d_L}$. Solving for $R_E$, $\boxed{R_E = 0.056pc = 1.7 \times 10^{15}\text{m}}$. Solving for $R_S = R_E^2/d_L = 1.2 \times 10^{10}$m. Using the relation between $R_S$ and $M$, $\boxed{M = 3.9 \times 10^6 M_\odot}$. This is close to the value derived from the orbits of Galactic center stars. So far microlensing by the Galactic center black hole has not been observed, but may be observable with the next generation of large ground-based telescopes.

d) (8 pt) A correlation between black hole mass, $M_{BH}$, and velocity dispersion of the bulges of spiral galaxies, $\sigma$, finds that $M_{BH} = 2 \times 10^8 M_\odot \left(\frac{\sigma}{200\text{kms}^{-1}}\right)^4$. For the Milky Way, **what bulge velocity dispersion** do you expect given the mass of our black hole? (If you did not complete part c, use $M_{BH} = 2 \times 10^6 M_\odot$ for the Milky Way).

The circular velocity of the bulge is $\sqrt{2}$ times larger than the velocity dispersion — **calculate** this value. **How does this compare** to the velocity of the local standard of rest, 220 km/s? **Explain** the apparent contradiction with the statement that the Milky Way has a nearly flat rotation curve.

Solution: Plugging in this value into the black hole mass - velocit dispersion relation, $\sigma = 200 \mathrm{kms}^{-1}(2 \times 10^6/2 \times 10^8)^{1/4}$, or $\boxed{\sigma = 63 \text{ km s}^{-1}}$. The corresponding circular velocity is $\boxed{v_c = \sqrt{2}\sigma = 89 \text{ km s}^{-1}}$. This is much smaller than the circular velocity at the local standard of rest. The statement that the Milky Way and other galaxies have flat rotation curves only applies to the disk portions of those galaxies - the bulges have rising rotation curves; hence the circular velocity is much smaller in the bulge/bar of the Milky Way.