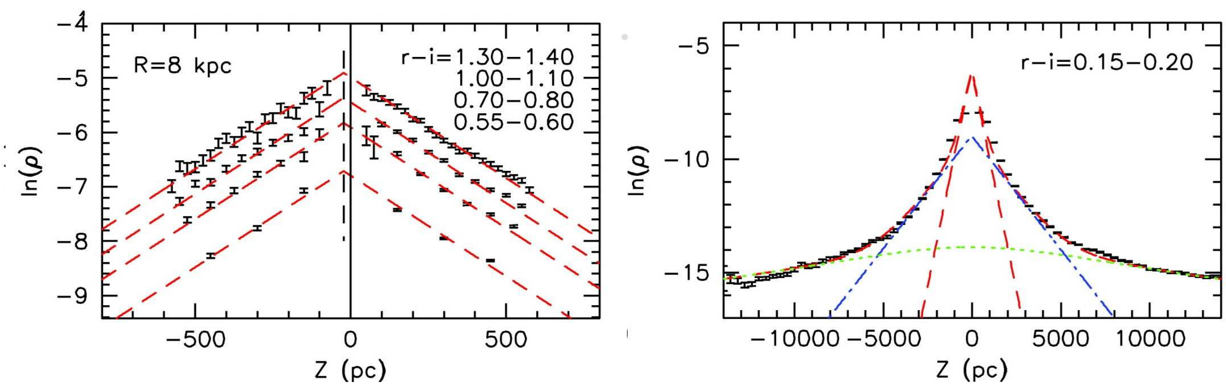
ASTR 323 Spring 2009

Midterm Solution

1. (28 pt) Milky Way Galaxy (5 parts, a-e)

   The following figure shows a plot of stellar number density, $\rho(Z)$ (stars/pc^3), as a function of height, Z (pc), below and above the location of the Sun in the Milky Way.

   Note: some of you had trouble deciphering the different lines (the dotted line was very faint in the photocopies, while the data looked similar to a dotted line). I have tried to take this into account in the grading, but please let me know if you feel you need a revised grade.



   a) (4 pt) **Why is** the peak of $\rho(Z)$ offset from $Z = 0$ in the left panel? **What is** the function, $\rho(Z)$, which is plotted as the dashed angled lines in the left panel (just give the functional form in terms of the mid-plane density, $\rho(0)$, and scale length, h_Z - you can ignore offset and assume it peaks at $Z = 0$)?

   Solution: The peak is offset as the Sun is slightly above the midplane of the Galactic disk, about 20 pc. Since the dashed lines are straight in a plot of $\ln\rho$ versus Z, and symmetric about $Z = 0$ (ignoring offset as instructed), this means $\ln(\rho) = \ln(\rho(0)) - |Z|/h_Z$, where h_Z is the inverse of the slope of the lines in this plot and has units of length, thus the term "scale length." So $\boxed{\rho(Z) = \rho(0)e^{-|Z|/h_Z}.}$

   b) (8 pt) **What are** the three different model components in the right panel (long dash, dash-dot, and dotted curves)? **Explain** why these have different spatial distributions.

   Solution: Long dash: thin disk; dash-dot: thick disk; dotted: halo. The velocity of stars increases in this order. Stellar populations with larger velocity dispersions in the Z direction can rise higher above the Galactic plane before falling back down into the Galaxy.

   c) (4 pt) **Compute** the surface density, Σ (stars per square parsec) of a disk with density in the vertical direction which scales as $\rho(Z)$ from part a, expressing your answer in terms of $\rho(0)$ and h_Z.

Solution: $\Sigma = \int_{-\infty}^{\infty} dZ\rho(Z) = \rho(0)\int_{-\infty}^{\infty} e^{-|Z|/h_Z}$. Since the density distribution is symmetric, we can integrate from 0 to ∞, substituting $x = Z/h_Z$, so $dZ = h_Z dx$, and then double our result: $\Sigma = 2\rho(0)h_Z \int_0^{\infty} e^{-x}dx$. Now, $\int_0^{\infty} e^{-x}dx = -e^{-x}|_0^{\infty} = 1$, so $\boxed{\Sigma = 2\rho(0)h_Z.}$

d) (10 pt) **Estimate** the scale lengths, h_z, of the two components with the highest mid-plane density from the right panel. **What is** the ratio of the densities of these at $Z = 0$? **What is** the ratio of their surface densities (from your result in part c)?

Solution: For the thin disk, the vertical axis goes from about -6.5 to -17, which is 9.5 scale heights, over 2900 pc, so $\boxed{h_{Z,thin} = 2900pc/9.5 = 305 \text{ pc}}$. For the thick disk, the vertical axis goes from -9 to -17, 8 scale heights, over 8000 pc, or $\boxed{h_{Z,thick} = 8000pc/8 = 1000 \text{ pc}}$. The ratio of the midplane densities is $\boxed{\rho_{thin}(0)/\rho_{thick}(0) = e^{-6.5-(-9)} = 12.}$ So, the ratio of the surface densities is $\boxed{\Sigma_{thin}/\Sigma_{thick} = 12\frac{305}{1000} = 3.7}$. Thus the thin disk has nearly four times as many stars as the thick disk (on the homework the ratio is about 3).

e) (2 pt) **What is** the observational evidence that our Galaxy contains a bar-shaped bulge?

Solution: The infrared light distribution measured by COBE towards the Galactic center shows a boxy shape to the isophotes which are brighter on one side than the other showing that the bulge is elongated, and the side that is closer appears larger and brighter causing the asymmetric brightness distribution (the tilt is about 20°). There are other pieces of evidence (not discussed in class) such as from microlensing studies of the Galactic bulge which indicates that the bulge needs to have a bar shape that is nearly pointed towards to explain the high probability of microlensing towards the bulge, and thus the large number of microlensing events that have been discovered.

2. (26 pt) Singular isothermal sphere (5 parts, a-e)

   A singular isothermal sphere has a density with respect to radius, R, which is given by: $\rho(R) = \frac{\sigma^2}{2\pi G R^2}$, where σ is the velocity dispersion.

   a) (4 pt) **Compute** the total mass contained with a sphere of radius R in terms of G, R and σ. **Check** that the dimensions of your answer are correct.

   Solution: $M(< R) = \int_0^R \rho(r) 4\pi r^2 dr = 2\sigma^2 G^{-1}\int_0^R dr$, so $\boxed{M = 2\sigma^2 R G^{-1}.}$ The dimensions of the right hand side are $(m/s)^2 m/(m^3 kg s^{-2}) = kg$, which has the correct dimensions of mass.

   b) (6 pt) **What is** the circular velocity, Θ (km/s), in terms of the enclosed mass, $M(< R)$? **Express** Θ in terms of σ. **Explain** the ratio between σ and Θ.

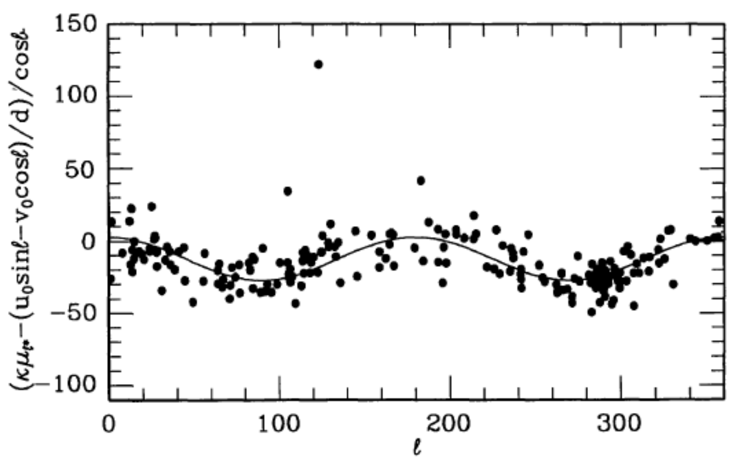
   Solution: Balancing centripetal acceleration and gravitational acceleration, $\frac{\Theta^2}{R} = \frac{GM(<R)}{R^2}$, so $\boxed{\Theta = \sqrt{\frac{GM}{R}}.}$ Plugging in the expression for $M(< R)$, $\Theta = \sqrt{2G\sigma^2 R/(GR)}$, so $\boxed{\Theta = 2^{1/2}\sigma,}$ which means Θ is constant. If all stars are on circular orbits in an isothermal sphere, then they will all have a velocity of Θ which balances centripetal acceleration versus gravity. If they are not on circular orbits, then the virial theorem states that the velocity dispersion should be determined by the gravitational potential; thus dimensionally $\sigma \sim \Theta$.

c) (4 pt) The Oort constants, A and B, are defined as $A = -\frac{1}{2}\left[\frac{d\Theta}{dR} - \Omega\right]$ and $B = -\frac{1}{2}\left[\frac{d\Theta}{dR} + \Omega\right]$, where $\Omega = \Theta/R$. **Express** A and B in terms of σ and R.

Solution: Well, $d\Theta/dR = 0$ since $\Theta =$ constant, since $\Omega = \Theta/R$, $\boxed{A = -B = 2^{-1/2}\sigma/R.}$

d) (6 pt) As discussed in class, the transverse velocity of stars in the plane of the disk of the Milky Way is given by $v_t = Adcos(2l) + Bd$ where d is the distance to a star and l is the Galactic longitude. Based on the following plot of v_t/d (km s^{-1} kpc^{-1}) versus Galactic longitude, l (°), **estimate** A and B at the Solar circle from this plot. Comparing to the expressions for A and B from part c, **is the rotation curve of the Milky Way flat** at the Solar circle?

Solution: The peak-to-trough amplitude is about 30 km/s/kpc, so $A \simeq 15$ km/s/kpc. The offset from zero is about -10, so $B \simeq -10$ km/s/kpc. When $d\Theta/dR = 0$, $A = -B$, so the rotation curve is not flat at the Solar circle. In fat, $d\Theta/dR = -(A+B) = -2.5$ km/s/kpc. So for every kpc further from the Galactic center, stars rotate slower by 2.5 km/s. Thus, the rotation curve is not precisely flat, but the gradient is small, so it is pretty close to being flat.



e) (6 pt) As discussed in class, double spiral arms can form when $\Omega_{lp} = \Omega - \kappa/2 \sim$ constant (i.e. independent of radius- "lp" stands for local pattern speed), where κ is the radial epicyclic frequency, $\kappa^2 = -4B\Omega$. **Compute** Ω_{lp} for an isothermal sphere in terms of Θ_0 and R. Given that Ω_{lp} is nearly constant in the disk of the Milky Way from about 5 to 10 kpc, while the disk has a nearly flat rotation curve, **what does this tell you** about the structure of the Milky Way?

Solution: $\kappa^2 = -4(-\frac{1}{2}\Omega)\Omega = 2\Omega^2$, so $\kappa = 2^{1/2}\Omega$. Thus, $\Omega_{lp} = \Omega - 2^{-1/2}\Omega = (1 - \frac{1}{\sqrt{2}})\Theta/R$. This means that Ω_{lp} decreases inversely with radius for an isothermal sphere - it is not constant. Now the Milky Way rotation curve is nearly constant which would imply that the density distribution is nearly that of an isothermal sphere if the Milky Way were spherically symmetric. However, this would imply that $\Omega_{lp} \propto R^{-1}$, while for the Milky Way $\Omega_{lp} \propto R^0$ (nearly). This seems to be a contradiction. The resolution is due to the fact that the Milky Way *is not spherically symmetric*. The disk has a different dependence of κ with radius than what we computed, and so Ω_{lp} can be nearly constant while Ω decreases inversely with radius.

3. (22 pt) Black holes (5 parts, a-e)

   a) (6 pt) **What are** the 3 characteristics of black holes in General Relativity? **Why do we think** only 2 are important for astrophysics?

   Solution: Black holes are characterized by their mass M, spin J, and charge Q. We think that charged black holes are quickly shorted out by attracting charges of the opposite sign, so only M and J are important for astrophysical black holes.

   b) (4 pt) If the speed of light in vacuum were half its value in another Universe, **how much larger or smaller** would the event horizon of a non-spinning black hole be of identical mass?

   Solution: Well, $R_s = GMc^{-2}$, so if c were cut in half, the event horizon would be four times larger. This is due to the fact that the escape velocity of the black hole is lower at larger radius, so if the speed of light were slower, light would be unable to escape from a larger radius.

   General relativity predicts a gravitational redshift, z, of an amount

$$1 + z = \frac{\lambda_1}{\lambda_2} = \left(1 - \frac{r_s}{r}\right)^{-1/2}, \tag{1}$$

   wher r_s is the Schwarzschild radius.

   Note: There was a typo in this problem - I left out a negative sign in front of the 1/2 *in the exponent on the exam - so I have tried to account for this in the grading of parts c and e. Please let me know if you think you need a re-grade.*

   c) (2 pt) **What do** λ_1 and λ_2 represent?

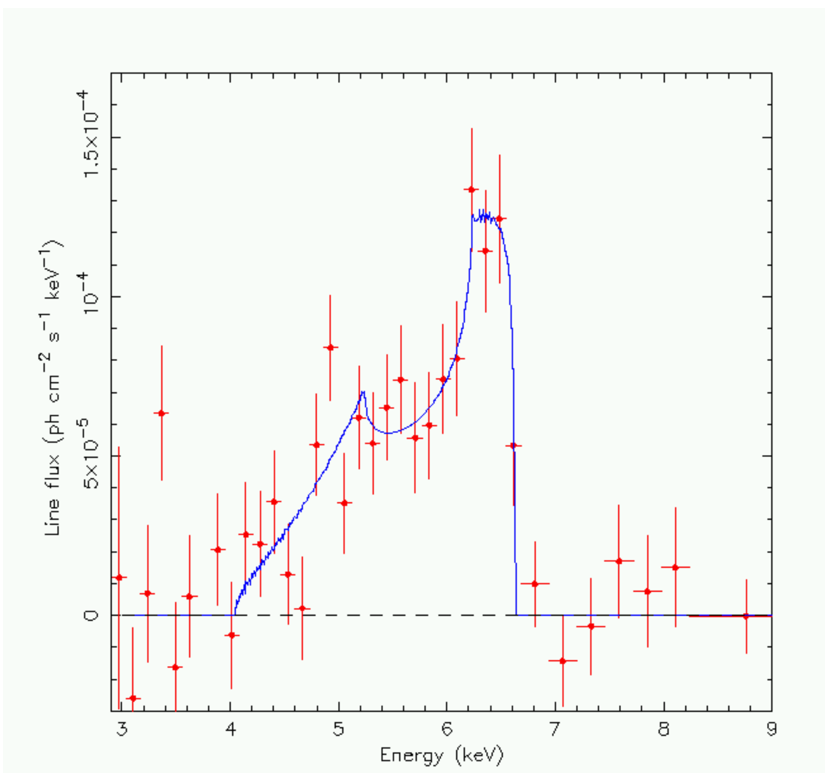
   Solution: λ_2 corresponds to the wavelength of the photon when it is emitted at radius r, while λ_1 is the wavelength seen by an observer at (near) $r = \infty$. Since $r_s < r$, the right hand side is always greater than one (with the correct sign in the exponent), so λ_1 ¿ λ_2 which is why this is called a redshift.

   d) (6 pt) **Estimate the density** of the Galactic Center black hole which has a mass of $4 \times 10^6 M_\odot$ ($M_\odot = 2 \times 10^{30}$ kg). If the Sun were to fall into this black hole, **would it be tidally disrupted** before reaching the event horizon? The radius of the Sun is $R_\odot = 7 \times 10^8$ m. **Explain** your reasoning.

   Solution: The Schwarzschild radius for the Galactic center black hole is $2GMc^{-2} = 1.2 \times 10^{10}$ m. The density of the Galactic center black hole is about $\rho_{BH} = 3M/(4\pi R_S^3) = 1.1 \times 10^6$ kg m^{-3}. The density of the Sun is $\rho_\odot = 3M_\odot/(4\pi R_\odot)^3 = 1400$ kg m^{-3}. Tidal disruption occurs at the event horizon when $\rho_{BH} > \rho_\odot$ (there is a factor of order unity that is omitted here), which means that the Sun would be tidally disrupted before falling into this black hole. The result of a tidal disruption could lead to gas orbiting the black hole at different rates, collision of the gas as it orbits, heating of the gas, and strong, transient emission.

e) (4 pt) The following figure shows the spectrum of an iron line emitted by a Seyfert galaxy, containing a supermassive black hole, intensity versus X-ray photon energy in keV. The iron line energy in the rest frame is 6.4 keV. The solid line shows a model for the iron line emitted from a disk orbiting a black hole. From the figure, **estimate the largest redshift**, z, in the model (hint: how does photon energy relate to wavelength)? If this is due to gravitational redshift, **what radius of emission** does this correspond to in units of r_s?

Solution: The lowest energy radiation emitted in the model is about 4 keV. Wavelength is inversely proportional to energy, so this means that the wavelength observed at infinity is 6.4/4 times larger than the emitted wavelength, corresponding to a redshift of $1+z = 6.4/4$, or $\boxed{z = 0.6.}$ Now, due to the error in the formula on the exam, this would corresond to an emission radius of $(6.4/4)^2 = 1 - r_s/r$, or $r = [1 - (6.4/4)^2]^{-1} = -0.64$ (clearly nonsense). With the correct formula one finds $r = [1 - (4/6.4)^2]^{-1} = 1.64$. This is *very* close to the event horizon of a black hole - just outside the photon circular orbit. Some astrophysicists have concluded that this black hole has to be spinning to have a disk with an inner edge close enough to the black hole to explain this large of a redshift.



4. (24 pt) Hubble tuning fork (3 parts, a-c)

a) (12 pt) The left hand side shows negative images of 3 galaxies. The right side shows their optical spectra. **Match** each galaxy image to its spectrum, and, more importantly, **explain why** you made these matches based on the morphology of the galaxies, spectral shape, and emission lines present in the spectra. If the left hand figures were color, **explain** what the color images would look like to the human eye.

Solution:

Note: Unfortunately the photocopies came out much worse than the printed version of the galaxy figures, so the morphologies of the galaxies were difficult to see. I have taken this into consideration when grading - please let me know if you need a re-grade.

1 - b - spiral galaxies have a mix of old and young stars; the old tend to be concentrated in the bulge (although they are also present in the disk and halo at lower density), and the young near the spiral arms. Thus the spectrum is intermediate between that of an irregular galaxy (only young stars) and an elliptical galaxy (only old stars), much like the middle plot. The spiral galaxy would appear blue near the spiral arms, with dark dust lanes, and red towards the center where the bulge lies.

2 - c - elliptical galaxies are primarily composed of old stars which have spectra dominated by cool red giants. These stars are redder and have strong absorption breaks near 4000 Å. In addition, the lack of young stars means that there is not strong photoionization, so the emission lines are weak. The image would appear reddish everywhere.

3 - a - irregular galaxies have strong star formation, which leads to lots of visible young stars which are very blue and hot enough to create strong emission lines, just as in the top spectrum. The image would appear blue everywhere from young stars.

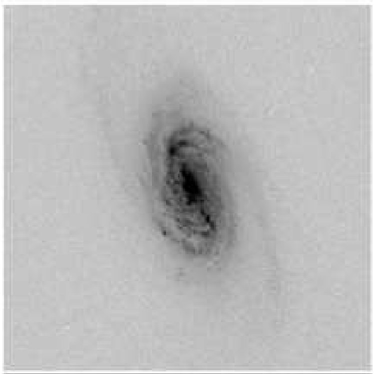
b) (6 pt) The Sersic intensity profile is given by $I = I_e e^{-(r/r_e)^{1/n} - 1}$. **What values** of n do you expect galaxies 1 and 2 to have? **Explain** your answer.

Solution: The top galaxy has a disk which typically has an exponential surface brightness profile with $n = 1$. The second galaxy is elliptical which typically has a de Vaucouleurs profile with $n = 4$. Some of you may point out that the bulge of the spiral should also have a de Vaucouleurs profile with $n = 4$.

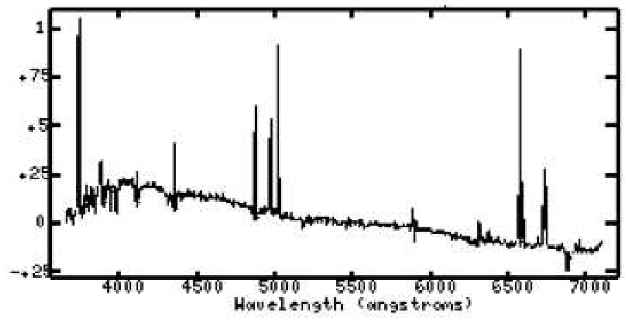
c) (6 pt) Taking an educated guess, **rank the ages** of the brightest stars in the three galaxies from least to greatest. Also taking an educated guess, **rank the total mass** of these three galaxies from least to greatest. **Explain** your answers

Solution: As mentioned above, the ages of the brightest stars should be ordered as 3, 1, 2; the same ordering applies for the masses of the galaxies.

1



a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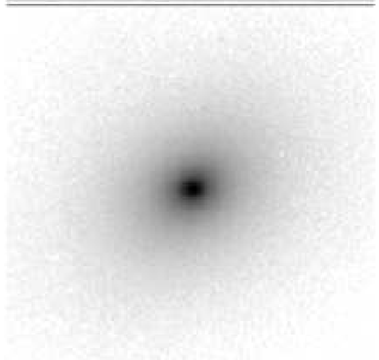
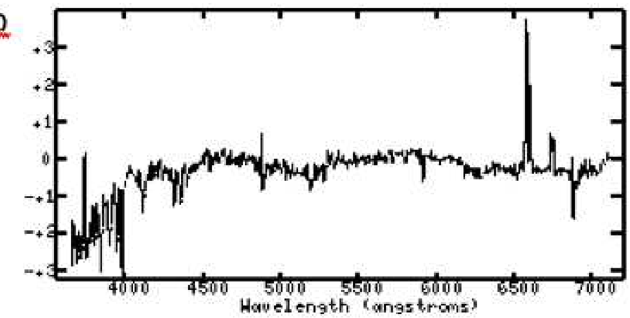
Wavelength (angstroms)

2



b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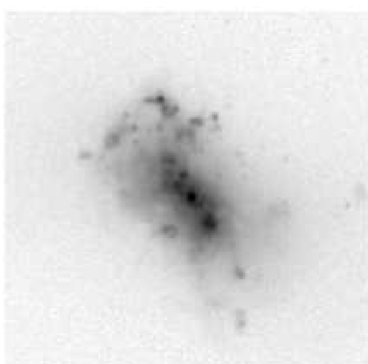
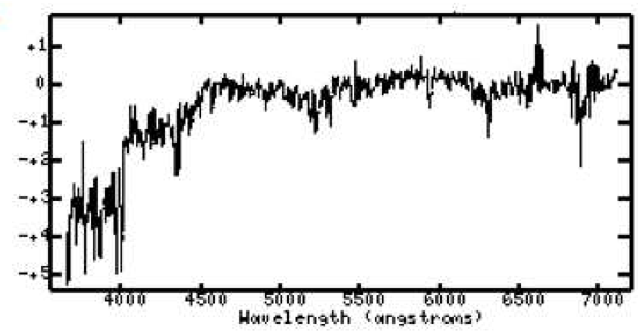
Wavelength (angstroms)

3



c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Wavelength (angstroms)