ASTR 323 Spring 2009

Problem Set 1 - Solution (55 points total)

Due: April 7, 2009
(CO = Carroll & Ostlie)

1. (5 pt) CO 24.9

   Solution: a) (3 pt) The thermal energy density is given by $u = \frac{3}{2}Nk_BT$ where N is the number density and k_B is the Boltzmann constant. The volume of the gas is $V = \pi HR^2 = \pi 8^2 0.16$ kpc^3 = 32 kpc^3. The total mass is $M = 10^{10}M_\odot = m_H V n_H$, where $N = n_H$ is the number density of atomic hydrogen and m_H is the mass of a hydrogen molecule. So the energy density is $u = \frac{3}{2}Mk_BT/(Vm_H) = 2 \times 10^{-15}$ Joule m^{-3}.

   b) (2 pt) The energy density in magnetic field is $u_B = 6 \times 10^{-14}$ Joule m^{-3}. This is 30 times larger than the energy density in atomic hydrogen, and thus will play an important role in the structure of the interstellar medium. Pressure scales with energy density, so the strong magnetic pressure can result in force gradients which are large. However, neutral gas does not interact with magnetic field, except via collisions with charged particles (e.g. cosmic rays), so this reduces the impact of the magnetic field on the neutral gas, but it can have a strong impact on the ionized gas. In some cases, though, the collisions between charged and neutral particles are frequent enough that the magnetic field can affect the neutral gas - this is important in star formation. In class I stated that the temperature of atomic hydrogen is more typically 10^2 K (molecular hydrogen has typical temperatures around 15 K, as given in this problem), in which case its pressure would be closer to that of the magnetic field. In any case the magnetic field does not impact (significantly) the stellar density structure or the dark matter structure.

2. (8 pt) Derive the expression for the enhancement in collision cross section due to gravitational focusing mentioned in class. Consider a small point mass approaching a star of mass M and radius R. When the separation is large, their relative velocity is v_∞ and the impact parameter

is b_∞. Due to gravity the point mass is attracted towards the star and grazes the surface of the star with a velocity v_{max}.

a) (2 pt) Use energy conservation to show that $v_\infty^2 = v_{max}^2 - v_{esc}^2$, where $v_{esc} = (2GM/R)^{1/2}$.

Solution: The energy at infinity is purely kinetic, given by $\frac{1}{2}mv_\infty^2$, where m is the (small) mass of the test particle. At impact, the energy is $\frac{1}{2}mv_{max}^2 - GMm/R$. Equating these and canceling $m/2$ gives: $v_\infty^2 = v_{max}^2 - 2GM/R = v_{max}^2 - v_{esc}^2$.

b) (2 pt) Show that angular momentum conservation implies $bv_\infty = Rv_{max}$.

Solution: At both points the velocity has no radial component, so the angular momentum is $\mathbf{L} = m\mathbf{v} \times \mathbf{r} = mvr\hat{z} = mbv_\infty\hat{z} = mRv_{max}\hat{z}$, where $\hat{z}$ is a unit vector perpendicular to the plane of scattering (chosen to have the correct handedness). Canceling $m\hat{z}$ gives the desired result.

c) (2 pt) Combine these results to prove that the collision cross section increases by a factor of $1 + v_{esc}^2/v_\infty^2$ over the geometric cross section of the star, as discussed in class.

Solution: Combining these two equations gives: $v_\infty^2 = v_\infty^2(b/R)^2 - v_{esc}^2$. Solving for b^2 gives $b^2 = R^2\left[1 + (v_{esc}/v_\infty)^2\right]$. The cross section for collision is just $\sigma = \pi b^2 = \pi R^2\left[1 + (v_{esc}/v_\infty)^2\right]$. The same enhancement factor applies for the collision between two identically sized stars.

d) (2 pt) How does the ratio of the collision cross sections for our Sun and a solar mass giant star which is 1 AU in size when $v_\infty = 10$ km/s compare to the ratio of their radii? Explain your result.

Solution: The giant star is 214 times larger than the Sun, so the geometric cross section is 46000 times larger. The enhancement factor for the Sun is 954, while for the giant star is only 5. Consequently the collision cross section is only 262 times larger for the giant star than for the Sun. The reason is that even though the giant star presents a larger target, as the impact parameter becomes larger, then gravitational focusing is weaker, so a colliding star starting with 16 times larger impact parameter will hit the giant star.

3. (8 pt) A star is found orbiting the black hole at the Galactic Center on a circular orbit which has a period of $P = 15.86 \pm 0.3$ years, an orbital

axis ratio of $\beta/\alpha = 0.702 \pm 0.016$, an angular size of the long axis of the orbit of $\alpha = 126.5 \pm 3.4$ mas, and a radial velocity at maximum separation on the sky of $v_r = 1357 \pm 20$ km/s (for the definition of these quantities see the diagram in Lecture 1).

a) (4 pt) Derive the distance to the Galactic Center, R_{GC}, in terms of α, P, v_r, and β/α using the expressions given in class. Using the error propagation formula, determine the uncertainty on R_{GC} in terms of the uncertainties of the other four quantities.

Solution: Combining the equations from slide 13, $R_{GC} = r/\alpha = \frac{Pv_r}{2\pi\alpha \sin i}$. Now, $\sin i = \sqrt{1 - \cos^2 i} = \sqrt{1 - (\beta/\alpha)^2}$. So the final result is:

$$R_{GC} = \frac{Pv_r}{2\pi\alpha\sqrt{1 - (\beta/\alpha)^2}}. \tag{1}$$

Propagating errors:

$$\frac{\sigma^2_{R_{GC}}}{R^2_{GC}} = \frac{\sigma^2_P}{P^2} + \frac{\sigma^2_\alpha}{\alpha^2} + \frac{\sigma^2_{v_r}}{v_r^2} + \frac{\mu^2}{(1-\mu^2)^2}\frac{\sigma^2_\mu}{\mu^2}, \tag{2}$$

where $\mu = \cos i = \beta/\alpha$.

b) (2 pt) Compute the distance to the center of the Galaxy using the values given in this problem, including the uncertainty. (Note: your computed uncertainty is actually smaller than the value in Ghez et al. 2008 due to your assuming a circular orbit and assuming that the black hole is stationary.)

Solution: Plugging in the numbers gives $R_{GC} = 8.04$ kpc (note that α needs conversion to radians and v_r needs conversion to parsec per year) and $\sigma^2_{R_{GC}}/R^2_{GC} = (.3/15.86)^2 + (3.4/126.5)^2 + (20/1357)^2 + .702^2/(1 - .702^2)^2(0.016/.702)^2 = 0.048^2$, yielding $\sigma_{R_{GC}} = 0.385$ kpc.

c) (2 pt) Which measured quantity needs to be improved upon most to reduce the uncertainty in distance to the Galactic Center? How do you think that improvement could best be achieved?

Solution: Well, the fractional uncertainties on P, α, v_r and μ are 0.019, 0.027, 0.015, 0.023. So the main improvement needs to be made in the astrometric precision. Generally astrometric precision scales with the diameter of the telescope (or baseline of an interferometer), so bigger telescopes will lead to an improvement. The VLT has been used as an interferometer which has a baseline of 85 meters, so this will lead to a great improvement if it has the sensitivity to follow the stars at the Galactic center. The interesting thing, though, is that all the fractional errors are nearly the same. The period accuracy will improve with a longer baseline of observations - this could take decades. The radial velocity precision will improve with higher signal-to-noise which requires larger telescopes with more collecting area. The error on μ also scales with astrometric error, so it will have similar improvements as α. In reality these stars are on highly eccentric orbits, and it turns out the astrometric measurements are limited by confusion with other stars when the orbiting star is near peribothon (pericentre for a black hole), so higher resolution observations will help with this as well.

4. (4 pt) Radio VLBI measurements of the position of the Galactic Center black hole reveal its motion on the sky with respect to distant, stationary background quasars of 6.063 ± 0.024 mas yr^{-1} (from the perspective of an observer at the local standard of rest, LSR). Assume for the moment that the radio source is precisely at the location of the center of the Galaxy and does not move. Using the distance to the Galactic center you computed in the previous problem, calulate the velocity of the LSR, as well as the uncertainty on your estimate.

   Solution: Well, if the black hole is stationary at the center of the galaxy, then the LSR is moving with a velocity of 6.6063 mas yr$^{-1} \times R_{GC} = 230$ km/s. The uncertainty on this value is $\sqrt{(0.024/6.603)^2 + (0.385/8.04)^2}$ $\times 230 = 11.07$ km s^{-1}.

5. (3 pt) CO 24.1 (You can use your results from the last two problems and assume that the Sun is at rest with respect to the LSR).

   Solution: The Sun has been around about 4.5 Gyr. The period of orbit of the LSR is $2\pi R_{GC}/v_{LSR} = 214$ Myr. Thus the Sun has gone around about 21 times (it has just reached drinking age in Solar Galactic years).

6. (4 pt) From the parameters given in class based on Juric et al. (2008), compute the surface number density of stars from the thin disk, the thick disk, and the halo at the solar circle (use the number density given in the discussion of collisions with the local number density ratios

given in the Juric model). What are the approximate ratios of these surface densities?

Solution: The surface density, Σ, is given by:

$$\Sigma = \int_{-\infty}^{\infty} \rho(z) dz. \tag{3}$$

If density has an exponential profile, then $\Sigma = 2H\rho(0)$. From the parameters given in lecture for the solar circle, $\Sigma_{thin} = 2 \times 300\text{pc} \times 0.085\text{pc}^{-3} \times \frac{1}{1+0.12+0.005} = 45$ stars pc^{-2}, $\Sigma_{thick} = 16$ stars pc^{-2}.

The halo star density needs to be integrated numerically:

$$\Sigma_{halo} = \rho_{halo} R_0 \int_{\infty}^{\infty} dx (1 + (x/q)^2)^{n/2}, \tag{4}$$

where $x = z/R_0$. This gives $\Sigma_{halo} = 1.38 \rho_{halo} R_0 = 4.2$ stars pc^{-2}.

So the ratios of the surface densities are approximately 10:3:1.

7. (4 pt) CO 24.19

   Solution: (a) (2 pt) From equation 24.44, $d\Theta/dR = -2.4$ km/s/kpc. This means that the circular velocity is declining the vicinity of the Solar circle, which can be seen in Figure 24.25. (b) (2 pt) Then $d\Theta/dR = 0$, and the rotation curve would be precisely flat at the solar circle (although it could have higher order curvature).

8. (5 pt) CO 24.24 How do your derived dark-matter mass density compare to the local stellar mass density?

   Solution: Integrating the density of the NFW halo outwards,

$$M(< r) = 4\pi \rho_0 a^3 \left[\frac{1}{1+x} + \ln(1+x) - 1 \right], \tag{5}$$

   where $x = r/a$. The mass is given at two radii, 50 and 230 kpc, just after equation 24.15, which is enough to solve for ρ_0 and a. The way I accomplished this was to plot $4\pi\rho_0 a^3$ versus a computed from the two radii: $\rho_0 a^3 = M(< 50kpc)/[(1+x_1)^{-1} + \ln(1+x_1) - 1]$ and $\rho_0 a^3 = M(< 230pc)/[(1+x_2)^{-1} + \ln(1+x_2) - 1]$, where $x_1 = 50/a$ and $x_2 = 230/a$. These two should be equal at the correct value of a, which I find to be $a = 28.89205$ kpc. Using this value of a, I find $\rho_0 = 0.0445 M_\odot$ pc^{-3}. At the solar circle, this gives a density of $0.01 M_\odot$ pc^{-3}. This is about 1/5 of the local stellar mass density in the disk.

9. (14 pt) CO 24.27

Solution: (a) (3 pt) Equation 24.56 yields: $2gA = -8\pi GAz\rho$. Canceling $2A$ gives $g = -4\pi G\rho z$ - the magnitude of gravitational acceleration increases linearly with $|z|$ and it is always directed towards the mid-plane. (b) (2 pt) The gravitational acceleration acting on a test particle gives $d^2z/dt^2 = g$, or $d^2z/dt^2 + kz = 0$, where $k = 4\pi G\rho$. (c) (3 pt) Well, the solution to this equation is $z = z_{max}\sin(\omega t + \phi)$, where $\omega = \sqrt{k} = \sqrt{4\pi G\rho}$. Differentiating with respect to time to get the vertical velocity, $w = dz/dt = z_{max}\omega\cos(\omega t + \phi)$. (d) (2 pt) The oscillation frequency is $\omega = 3 \times 10^{-15}$ radians/sec. This corresponds to $P = 2\pi/\omega = 66$ Myr. (e) (2 pt) Taking $t = 0$ now, $\tan\phi = \omega z_{\odot}/w_{\odot} = 0.308$, or $\phi = 17°$. Then, plugging this in to the expression for z and solving for z_{max} gives: $z_{max} = 79$ pc. This is 26% of one scale height of the thin disk, so the Sun has a relatively small vertical motion compared to most disk stars. (f) (2 pt) Well, from problem 24.1 we found a period of 214 Myr which means 3.24 oscillations per orbit.