

# ASTR 323 Spring 2009

## Problem Set 2 - Solution (20 pt)

Due: April 14, 2009

(CO = Carroll & Ostlie)

1. (4 pt) Based on Lecture 1 and results from your problem 3 of homework 1, what is the mass of the Galactic Center black hole and uncertainty on the mass?

Solution: Based on the equations from slide 13 of lecture 1, we want to solve for  $M_{BH}$  in terms of observable quantities,  $v_r$ ,  $\alpha$ ,  $P$ , and  $\beta/\alpha$ . The radius,  $r$ , can be eliminated by combining equations 1 and 3, giving  $M_{BH} = \frac{Pv_r^3}{2\pi G \sin^3 i}$ . As before,  $\sin i = \sqrt{1 - \beta^2/\alpha^2}$ , so the final result is (2 pt)

$$M_{BH} = \frac{Pv_r^3}{2\pi G} (1 - \beta^2/\alpha^2)^{-3/2}. \quad (1)$$

Plugging in the numbers given in the first homework,  $M_{BH} = 4.13 \times 10^6 M_\odot$ .

Error propagation yields (2 pt):

$$\frac{\sigma_{BH}^2}{M_{BH}^2} = \frac{\sigma_P^2}{P^2} + 9 \frac{\sigma_{v_r}^2}{v_r^2} + 9 \frac{\mu^2}{(1 - \mu^2)^2} \frac{\sigma_\mu^2}{\mu^2}, \quad (2)$$

where  $\mu = \cos i = \beta/\alpha$ .

This gives:

$$\frac{\sigma_{BH}^2}{M_{BH}^2} = (.3/15.86)^2 + 9(20/1357)^2 + 9 \frac{.702^2}{(1 - .702^2)^2} (0.016/.702)^2 = 0.106^2, \quad (3)$$

yielding  $M_{BH} = (4.13 \pm 0.44) \times 10^6 M_\odot$ .

2. (2 pt) CO 24.31

Solution: About  $5 \times 10^6 M_\odot$ . This is about the mass estimated in problem 1. As mentioned in class, the accretion rate right now is actually quite a bit lower than this average value.

3. (3 pt) CO 24.34

Solution: I actually prefer to begin with the equation just before 19.4, replacing the moon with the Sun and the planet with a black hole. The Roche limit is then  $r = \left(\frac{2M_{BH}}{M_{\odot}}\right)^{1/3} R_{\odot} = 1.4 \times 10^8$  km. The Schwarzschild radius is  $R_S = 2GMc^{-2} = 1.1 \times 10^7$  km, or about 10 times smaller. This means that a star like our Sun would be tidally disrupted before falling through the event horizon of a black hole of this mass, possibly causing a transient flare. Here is a recent paper on this topic:

<http://arxiv.org/pdf/0903.1107v1>

4. (6 pt) CO 17.9 (This problem relates to a recent paper which uses constraints on the size and luminosity of Sgr A\* to prove that it has an event horizon: <http://arxiv.org/abs/0903.1105>)

Solution:

a) (2 pt) The luminosity measured at a large distance,  $R_{\infty} \gg R_S$  (so that  $1 - R_S/R_{\infty} \sim 1$ ) from the black hole (“infinity”) is given by  $L_{\infty} = h\nu_{\infty}/\Delta t_{\infty}$  where  $h\nu_{\infty}$  is the average energy of a photon crossing the sphere of area  $4\pi R_{\infty}^2$  and  $\Delta t_{\infty}$  is the average time between photons. According to equation 17.13, this can be related to the luminosity,  $L$ , at a radius  $R$  as  $L_{\infty} = h\nu_0(1 - R_S/R)^{1/2}/[\Delta t_0/(1 - R_S/R)^{1/2}] = L(1 - R_S/R)$  where  $L \equiv h\nu_0/\Delta t_0$ , where  $h\nu_0$  is the average energy per photon leaving the sphere of radius  $R$  and  $t_0$  is the average time between photons leaving the sphere at radius  $R$ .

b) (2 pt) Well, since Wien’s Law applies for both observers,  $\lambda_{\infty}T_{\infty} = \lambda_0T_0$ . Since  $\lambda_{\infty} = \lambda_0/(1 - R_S/R)^{1/2}$ ,  $T_{\infty} = T_0(1 - R_S/R)^{1/2}$ . This can also be gotten by realizing that temperature is actually a unit of energy, and the energy which is being measured in Wien’s law is the energy of photons which are  $h\nu$ . So variation of the temperature of a blackbody distribution of photons with radius should have the same dependence as  $\nu$  in equation 17.13.

c) (2 pt) Applying the Stefan-Boltzmann law, each observer obtains or  $R \propto L^{1/2}/T_0^2$ . So,  $R_{\infty}/R = (L_{\infty}/L)^{1/2}(T_0/T_{\infty})^2 = (1 - R_S/R)^{1/2}/(1 - R_S/R) = (1 - R_S/R)^{-1/2}$ . This is a very small effect for main sequence stars or even white dwarfs, but it can be much larger for neutron stars

which have masses  $\sim 1.4M_{\odot}$  and sizes  $\sim 15$  km, so  $R_{\infty} \sim 1.09R$ , or a difference of 9%.

5. (2 pt) CO 17.10

Solution: The density of Earth is  $5.5 \text{ g cm}^{-3}$ , so a sphere of that diameter with the density of Earth has a mass of  $M = 1.2 \times 10^{41} \text{ g}$ . The escape velocity from such an object would be  $v = \sqrt{2GM/R} = 3 \times 10^{10} \text{ cm s}^{-1}$  which is the speed of light. Consequently light could not escape and the object would be a black hole.

6. (3 pt) CO 17.14

Solution: Take a rectangular patch at  $(\theta, \phi)$  with an area  $dA = \Delta\mathcal{L}_{\theta} \times \Delta\mathcal{L}_{\phi}$ , where  $\Delta\mathcal{L} = \sqrt{-(ds)^2}$  is defined by equation 17.18 and  $\Delta\mathcal{L}_{\theta}$  is the border along the  $\theta$  direction and  $\Delta\mathcal{L}_{\phi}$  is the border in the  $\phi$  direction. Then,  $\Delta\mathcal{L}_{\theta} = r d\theta$  since  $dt = dr = d\phi = 0$  in this direction, while  $\Delta\mathcal{L}_{\phi} = r \sin\theta d\phi$  since  $dt = dr = d\theta = 0$  in this direction. Now, we need to carry out an integral over the surface with  $dr = dt = 0$  at a radius  $r = R_S = 2GM/c^2$ ,

$$A = \int dA = \int \Delta\mathcal{L}_{\theta} \Delta\mathcal{L}_{\phi} = \int_0^{\pi} \int_0^{2\pi} R_S d\theta R_S \sin\theta d\phi = 4\pi R_S^2. \quad (4)$$