ASTR 323 Spring 2009

Problem Set 6 Solutions, 30 points total

Due: May 19, 2009

1. (4 pt) Measurements of the size of the quasar broad line regions indicate

$$\frac{R_{blr}}{10^{15}m} = 0.26\left(\frac{L_{bol}}{10^{37}W}\right)^{1/2}, \tag{1}$$

   where R_{blr} is the size of the broad line region.

   Assuming that all quasars shine at 10% of the Eddington limit and that the broad line region has a velocity dispersion that scales as $\sigma_{blr}^2 = 0.18GM_{BH}/R_{blr}$, derive the mass of the black hole as a function of quasar luminosity, derive the velocity dispersion as a function of quasar luminosity. For a typical quasar luminosity of 5×10^{39} W, what is the velocity dispersion of the broad line region? Why are these called "broad" lines?

   Solution: Okay - if black holes accrete at 10% of the Eddington luminosity, then $L_{bol} = 1.3 \times 10^{30}(M/M_{\odot})$ W, so $\boxed{M = (L_{bol}/1.3 \times 10^{30}\mathrm{W})M_{\odot}.}$ Thus, the velocity dispersion is $\sigma_{blr}^2 = 2.25 \times 10^{-7}\mathrm{m}^2/\mathrm{s}^2\left(\frac{L_{bol}}{\mathrm{W}}\right)^{1/2}$, or $\boxed{\sigma_{blr} = 4740\mathrm{km/s}(L_{bol}/10^{40}\mathrm{W})^{1/4}.}$

2. (2 pt) CO 28.6

   Solution: The efficiency is $\eta = \frac{R_S}{4R}$, so $\boxed{\eta = 1/12}$ for a nonrotating black hole, while $\boxed{\eta = 1/2}$ for a rotating black hole. This is remarkable - up to 50% of the energy may be extracted! (It's actually a bit less than than for a maximally rotating black hole due to capture of photons by the black hole and other general relativistic effects.)

3. (9 pt) 3C 273: The 3C 273 spectrum from Lecture 12 shows several Balmer lines marked. Balmer lines have wavelength of $\lambda = 912\text{Å}\left(\frac{1}{4} - \frac{1}{n^2}\right)^{-1}$, where $n > 2$ is the quantum number of the excited state of the Hydrogen atom before de-exciting.

a) (3 pt) From the ratios of the wavelengths of the lines in the figure, identify n for each line and estimate the redshift of the quasar.

Solution: Well, the lines are located at approximately $\lambda = 7700$, 5700, 5100, 4750 Å. The ratios of these wavelengths to the longest wavelength are 1, 0.74, 0.66, 0.62. The ratio of the first four Balmer wavelengths ($n = 3-6$) to the $n = 3$ wavelength is 1, (1/4-1/9)/(1/4-1/16) = 0.74, (1/4-1/9)/(1/4-1/25) = 0.66, and (1/4-1/9)/(1/4-1/36) = 0.625. This almost precisely matches the wavelength ratios of the four lines in the 3C 273 spectrum, so we can be fairly certain that these are the four longest wavelength Balmer series lines: $H\alpha, H\beta, H\delta$, and $H\gamma$. The redshifts are $z = 7700/912(1/4-1/9) - 1 = 0.17$, $z = 5700/912(1/4-1/16) - 1 = 0.17$, $z = 5100/912(1/4-1/25) - 1 = 0.17$, and $z = 4750/912(1/4-1/36) - 1 = 0.16$. So, the redshift is about $\boxed{z = 0.17}$. This is close to the widely accepted value of $z = 0.158$.

b) (2 pt) Supposing the optical flux of the quasar is $F_{opt} = 10^{-13}$ W m^{-2}, and that the bolometric luminosity of quasar is 9 times the optical luminosity, estimate the bolometric luminosity of 3C 273 (you may assume that the optical emission is isotroic).

Solution: Given this redshift, I estimate the distance to 3C 273 is about 689 Mpc ($h = 0.74$). So, the bolometric luminosity is about $\boxed{L = 5 \times 10^{39}\ \text{W} = 1.2 \times 10^{13} L_\odot.}$ This is *extremely* bright for such a compact source.

c) (1 pt) Estimate the size of the broad line region from the expression in problem 1.

Solution: This relation yields $\boxed{R_{BLR} = 5.6 \times 10^{15}\ \text{m}}$. This is about 745 times as large as the orbit of Pluto.

d) (1 pt) The width of the Balmer lines is about 3500 km/s. Estimate the mass of the central black hole.

Solution: Also using the expression from problem 1, I find $\boxed{M = 3 \times 10^9 M_\odot}$, very massive.

e) (1 pt) Estimate the Eddington luminosity of the black hole.

Solution: The Eddington luminosity is $L_{Edd} = 1.3 \times 10^{31} (M/M_\odot)$ W $= 4 \times 10^{40}$ W $= \boxed{10^{14} L_\odot.}$

f) (1 pt) Estimate Eddington ratio of 3C 273 (the ratio of the observed luminosity to the Eddington luminosity).

Solution: I find an Eddington ratio of $\boxed{12\%}$. This is a typical value (compare to figure 28.26 in the book).

4. (6 pt) CO 28.10

Solution: Let's call $\beta = v_{app}/c$. Then, for $v/c = 1$, $\beta = \sin\phi + \beta\cos\phi$. Writing this as $\beta(1-\cos\phi) = \sin\phi$, squaring gives: $\beta^2(1 - 2\cos\phi + \cos^2\phi) = 1 - \cos^2\phi$. Setting $x = \cos\phi$, $x^2 - 2\beta^2/(1+\beta^2)x + (\beta^2 - 1)/(\beta^2+1) = 0$. This has the solution:

$$\begin{aligned} x &= \beta^2/(1+\beta^2) \pm \sqrt{\beta^4/(1+\beta^2)^2 - (\beta^4-1)/(\beta^2+1)^2} \\ &= 1, (\beta^2-1)/(1+\beta^2). \end{aligned} \quad (2)$$

So, superluminal motion can occur for $\boxed{\frac{\beta^2-1}{\beta^2+1} < \cos\phi < 1}$ (equation 28.16).

Minimizing equation (28.15) with respect to ϕ,

$$d(v/c)/d\phi = \frac{\beta}{(\sin\phi + \beta\cos\phi)^2}(\cos\phi - \beta\sin\phi) = 0. \quad (3)$$

This is satisfied for $\cos\phi = \beta\sin\phi$ or $\boxed{\cot\phi = \beta}$ (equation 28.18). Now, $\cot^2\phi + 1 = \csc^2\phi$, so $\sin\phi = 1/\sqrt{\beta^2+1}$ and $\cos\phi = \beta/\sqrt{1+\beta^2}$. Plugging this into equation (28.15), $\frac{v_{min}}{c} = \frac{\beta\sqrt{\beta^2+1}}{1+\beta^2}$, or $\boxed{\frac{v_{min}}{c} = \sqrt{\frac{\beta^2}{1+\beta^2}}}$ (equation 28.17).

What is the minimum value of jet speed necessary to see superluminal motion?

Solution: For superluminal motion, $\beta > 1$, so $\boxed{v_{min}/c > \sqrt{\frac{1}{2}}.}$ Thus a jet must be going at about 70% of the speed of light to be apparent as superluminal for observers close enough to the jet axis.

5. (5 pt) CO 28.15

Solution: Using the small angle formula ($\sin x \sim x$), $\theta d_S = \beta d_S + \phi(d_S - d_L)$. Now, $r_0 = d_L\theta$, so $\phi = \frac{4GM}{\theta d_L c^2}$, so:

$$\theta d_S = \beta d_S + \frac{4GM}{\theta d_L c^2}(d_S - d_L). \quad (4)$$

Multiplying through by θ/d_S on both sides and rearranging gives: $\boxed{\theta^2 - \beta\theta - \frac{4GM(d_S - d_L)}{d_L d_S c^2} = 0.}$

The solution to the quadratic equation $\theta^2 - b\theta + c = 0$ can be written as $(\theta - \theta_1)(\theta - \theta_2) = \theta^2 - (\theta_1 + \theta_2)\theta + \theta_1\theta_2 = 0$.

We can thus immediately identify $\boxed{\theta_1 + \theta_2 = \beta}$ and $\theta_1\theta_2 = -\frac{4GM(d_S - d_L)}{d_L d_S c^2}$, or $\boxed{M = -\frac{\theta_1\theta_2 d_L d_S c^2}{4G(d_S - d_L)}.}$

6. (4 pt) CO 28.16

Solution: Using Hubble's law, $d_S = 11$ Gpc and $d_L = 2$ Gpc (we will learn more accurate formulas later). From the problem, $\theta_1\theta_2 = 0.8''(0.8'' - 2.22'') = -2.6 \times 10^{-11}$ rad^2. Plugging this into the equation we just derived, $\boxed{M = 3 \times 10^{11} M_\odot.}$ This is fairly similar to the result obtained with more careful computations.