ASTR 323 Spring 2009

Problem Set 7 solution - 16 points total

1. (3 pt) CO 29.10

   Solution: For a non-flat Universe the Friedmann equation is:

   $$(H^2 - 8\pi G\rho/3)R^2 = -kc^2. \tag{1}$$

   Setting $\rho_c = 3H^2/(8\pi G)$ and dividing this equation by H^2R^2

   $$1 - \rho/\rho_c = 1 - \Omega = -kc^2(1+z)^2/H^2, \tag{2}$$

   using $R = 1/(1+z)$. Multiplying this by $(1+z)/\Omega$ and applying equation 29.23:

   $$(1/\Omega - 1)(1+z) = -kc^2(1+z)^3/(\Omega H^2) = -kc^2(1+z)^3/[(1+z)^3\Omega_0 H_0^2]. \tag{3}$$

   The right hand side of this equation is equation is a constant, so that means the left hand side is constant. At $t = t_0$ (now), the left hand side is $1/\Omega_0 - 1$, so

   $$1/\Omega - 1 = (1/\Omega_0 - 1)/(1+z). \tag{4}$$

   This means that as z increases, $1/\Omega - 1$ goes to zero. That means that a pressureless, dust Universe becomes nearly flat at very high redshift (or early times).

2. (3 pt) CO 29.12

   Solution: Multiplying equation 29.10 by R and taking the time derivative:

   $$\dot{R}^3 + 2R\dot{R}\ddot{R} - \frac{8\pi G}{3}\frac{d(R^3\rho)}{dt} = -kc^2\dot{R}. \tag{5}$$

   Inserting equation 29.50:

   $$\dot{R}^3 + 2R\dot{R}\ddot{R} + \frac{8\pi G}{3}\frac{P}{c^2}3R^2\dot{R} = -kc^2\dot{R}. \tag{6}$$

   Using equation 29.10 to eliminate $-kc^2$:

   $$\dot{R}^3 + 2R\dot{R}\ddot{R} + \frac{8\pi G}{3}\frac{P}{c^2}3R^2\dot{R} = (\dot{R}^2 - 8\pi G\rho/3R^3)\dot{R}. \tag{7}$$

We can cancel the $\dot{R}^3$ on both sides of the equation and then divide by $2R\dot{R}$ and rearrange to find: $\boxed{\ddot{R} + \frac{4\pi G R}{3}\left[\rho + \frac{3P}{c^2}\right] = 0.}$

3. (3 pt) Just as the universe has a cosmic microwave background dating back to the time when the universe was opaque to photons, it has a cosmic neutrino background dating back to the earlier time when the universe was opaque to neutrinos. The calculated number density of cosmic neutrinos is $n_\nu = 3.36 \times 10^8$ m^{-3}.

   a) (1 pt) How many cosmic neutrinos are inside your body right now?

   Solution: I am 100 kg (please don't convert that to pounds) and I am mostly water which weighs 1000 kg m^{-3}, so my volume is about 0.1 m^{-3}. Multiplying this by the neutrino density I find that there are about 3.36×10^7 cosmic neutrinos in my body. These interact only very weakly so there is no harm done to my body.

   b) (2 pt) What average neutrino mass, m_ν, would be required to make the mass density of the cosmic neutrinos equal to the critical density? Express your answer in eV c^{-2}. How does this compare to the upper limit on the mass of neutrinos of < 2 eV c^{-2}?

   Solution: Taking the closure density of the Universe, 10^{-26} kg m^{-3}, and dividing by the number of neutrinos per cubic volume, the average mass per neutrino would have to be 3×10^{-35} kg to create the closure density of the Universe. This corresponds to about $\boxed{1.7 \times 10^5 \text{ eV c}^{-2}}$ (this is about one third of the mass of the electron). This is about 100,000 times larger than the mass limits on neutrinos, 2 eV c^{-2}, derived from particle physics experiments (as well as some other astrophysics constraints). Consequently neutrinos only contribute a very small fraction to the energy density of the Universe.

4. (3 pt) CO 29.15. Based on your answer, what is w for the cosmological constant "fluid" for which $\rho_\Lambda(t) = \rho_{\Lambda,0}$ (i.e. the energy density is independent of time)?

   Solution: Starting from cosmological version of the first law of thermodynamics and plugging in the equation of state $P = w\rho c^2$,

$$\frac{d(R^3\rho)}{dt} = -w\rho\frac{dR^3}{dt}. \tag{8}$$

Divide both sides by ρR^3:

$$\frac{1}{\rho R^3}\frac{d(\rho R^3)}{dt} = -w\frac{1}{R^3}\frac{dR^3}{dt}. \tag{9}$$

Now, $(1/x)dx/dt = d(\ln x)/dt$, so

$$\frac{d\ln(\rho R^3)}{dt} = -w\frac{d\ln(R^3)}{dt}. \tag{10}$$

Now, since w is a constant, we can insert it into the differential. Since $-w\ln(R^3) = \ln(R^3)^{-w} = \ln(R^{-3w})$, we get

$$\frac{d\ln(\rho R^3)}{dt} = \frac{d\ln(R^{-3w})}{dt}. \tag{11}$$

Since both sides are perfect differentials, we can integrate with respect to time (adding a constant of integration) to find:

$$ln(\rho R^3) = \ln(R^{-3w}) + \ln\rho_0, \tag{12}$$

where I have cleverly chosen the constant of integration so that when $R = 1$, $\rho = \rho_0$, the current density. Taking the exponential of this equation,

$$\rho R^3 = \rho_0 R^{-3w}, \tag{13}$$

or $\boxed{\rho = \rho_0 R^{-3(1+w)}}$.

Now, for the cosmological constnat, $\rho_\Lambda \propto R^0$, so $-3(1+w) = 0$, or $w = -1$. This is a **very** strange equation of state: $P_\Lambda = -\rho_\Lambda c^2$. What is meant by negative pressure? Well, if you put fluid with positive pressure in a box, the fluid tries to press outwards - pressure is force divided by area, so this tells you that there is an outwards force for every bit of surface area of the box. Thus, negative pressure would lead to an *inward* force on every bit of surface of the box. That's weird. Even weirder is that you may point out that this doesn't make sense - the cosmological constant is what is supposed to be *accelerating* the Universe. But the fluid in a box does not take into account the *gravitational* effect of negative pressure. Insert the equation of state for the cosmological constant, $P_\Lambda = -\rho_\Lambda c^2$ in our result from the second problem, and we find

$$\ddot{R} = \frac{8\pi GR}{3c^2}\rho_\Lambda. \tag{14}$$

Thus the negative pressure results in a *positive* acceleration of the Universe. The solution of this equation is an exponentially growing scale factor as a function of time! Very weird indeed. The way to interpret this is that the negative pressure acts like a negative energy density in the acceleration equation so that the gravitational effect of the negative pressure is 3 times as strong as the energy density of the cosmological constant, so this effectively causes repulsive gravity, causing the scale factor to *accelerate* its expansion, hence the positive value of $\ddot{R}$.

5. (4 pt) A universe with $\rho_\Lambda = k = 0$ with no radiation has a Friedmann equation:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho_{c,0}}{3}\frac{1}{R^3}, \tag{15}$$

where $\rho_{c,0}$ is the current critical density.

a) (2 pt) What is the function form of $R(t)$ given that $R = 1$ at $t = t_0$?

Solution: Well, multiplying by R^2 and taking the square root of this equation: $\dot{R} = H_0 R^{-1/2}$ where $H_0 = [8\pi G\rho_{c,0}/3]^{1/2}$ (I know this is true since at $t = t_0$ when $R = 1$, $H_0 \equiv \dot{R}(t_0)$). Integrating this equation,

$$H_0 \int_0^t dt' = H_0 t = \int_0^R dR' R'^{1/2} = \frac{2}{3}R^{3/2}. \tag{16}$$

Solving for $R(t)$, $R(t) = (3H_0 t/2)^{2/3}$. But this means that at $t = t_0$, $R(t) = 1 = (3H_0 t_0/2)^2$, so $t_0 = 2/(3H_0)$.

b) (1 pt) What is t_0 in terms of the Hubble constant H_0?

Solution: As just said, $t_0 = 2/(3H_0)$.

c) (1 pt) In our universe $H_0 = 70$ km s^{-1} Mpc^{-1} and the oldest stars have an age of $t_* = 13$ Gyr. Are these two observations consistent with a flat universe with $\rho_0 = \rho_{c,0}$?

Solution: Well, with this H_0, $t_0 = 9.3$ Gyr which is younger than the age of the oldest stars. Thus a flat Universe in which the matter density equals the critical density is ruled out by the observations.